Probability and Measure 2021-2022
Exam 1 Solutions

1. Let A be a set in the Borel σ-algebra of $\mathbb{R}$.

    (a) Prove that the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = m(A \cap (-\infty, x])$ is continuous (m denotes Lebesgue measure).

    (b) Assume that A has positive Lebesgue measure. Prove that for every $\delta > 0$, there exists $x \in \mathbb{R}$ such that $m(A \cap (x, x + \delta)) > 0$.

    Solution.

    (a) For any $x \in \mathbb{R}$ and $s > 0$ we have

    $$F(x + s) - F(x) = m(A \cap (-\infty, x + s]) - m(A \cap (-\infty, x]) = m(A \cap (x, x + s]) \le m((x, x + s]) = s,$$

    so F is Lipschitz continuous, so it is continuous.

    (b) Fix $\delta > 0$. If we had $m(A \cap (x, x + \delta)) = 0$ for all x, we would have

    $$m(A) \le \sum_{k \in \frac{\delta}{2}\mathbb{Z}} m(A \cap (k, k + \delta)) = 0,$$

    so $m(A) = 0$, contradicting the assumption that $m(A) > 0$.

2. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an integrable function (with respect to the measure space $(\mathbb{R}^d, \mathcal{B}^d, m)$). Let $E_1, E_2, \ldots$ be Borel subsets of $\mathbb{R}^d$ with the property that every $x \in \mathbb{R}^d$ belongs to at most finitely many of these sets. Prove that

    $$\lim_{n \to \infty} \int_{E_n} f \; \mathrm{d}m = 0.$$

    Solution. The assumption implies that the functions $f_n := f \cdot \mathbb{1}_{E_n}$ converge to zero pointwise. Moreover, these functions satisfy $|f_n| \le |f|$. Since $\int_{\mathbb{R}^d} |f| \mathrm{d}m < \infty$, the Dominated Convergence Theorem implies that

    $$\int_{E_n} f \; \mathrm{d}m = \int_{\mathbb{R}^d} f_n \; \mathrm{d}m \xrightarrow{n \to \infty} 0.$$

3. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. Assume that $\mathcal{F}_2$ is the trivial σ-algebra, that is, $\mathcal{F}_2 = \{\Omega_2, \varnothing\}$. Prove that if $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is $(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{B})$-measurable, then $f(\omega_1, \omega_2)$ does not depend on ω_2 (that is, $f(\omega_1, \omega_2) = f(\omega_1, \omega_2')$ for all ω_1, ω_2, ω_2'.

    Solution. If the statement were false, there would exist $\omega_1 \in \Omega_1$ and $\omega_2, \omega_2' \in \Omega_2$ such that $f(\omega_1, \omega_2) \neq f(\omega_1, \omega_2')$. Letting $x := f(\omega_1, \omega_2)$, this would imply that $\omega_2 \in (f_{\omega_1})^{-1}(\{x\})$, and $\omega_2' \notin (f_{\omega_1})^{-1}(\{x\})$. In particular $(f_{\omega_1})^{-1}(\{x\}) \neq \varnothing$ and $(f_{\omega_1})^{-1}(\{x\}) \neq \Omega_2$, so $(f_{\omega_1})^{-1}(\{x\}) \notin \mathcal{F}_2$. This implies that f_{ω_1} is not $(\mathcal{F}_2, \mathcal{B})$-measurable, which is a contradiction because f is $(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{B})$-measurable and sections of measurable functions are measurable.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Give the definition of the set $\mathcal{L}^\infty(\Omega, \mathcal{A}, \mu)$, of the semi-norm $\| \cdot \|_\infty$ and of the set $L^\infty(\Omega, \mathcal{A}, \mu)$.

    Solution.

    $$\mathcal{L}^\infty(\Omega, \mathcal{A}, \mu) := \{f : \Omega \to \bar{\mathbb{R}} : f \text{ is measurable and there exists } C < \infty \text{ such that } |f| \le C \text{ almost everywhere}\}.$$

    For each $f \in \mathcal{L}^\infty(\Omega, \mathcal{A}, \mu)$, we let $\|f\|_\infty := \inf\{C : |f| \le C \text{ almost everywhere}\}$. Finally, $L^\infty(\Omega, \mathcal{A}, \mu)$ is defined as the quotient space $\mathcal{L}^\infty(\Omega, \mathcal{A}, \mu)/\mathcal{N}$, where $\mathcal{N}$ is the set of functions f with $\|f\|_\infty = 0$.